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STATISTICAL METHOD TO EVALUATE CONVERGENCE OF NON-LINEAR OPTIMIZATION ALGORITHMS IN CALL CENTERS PROBLEMS

Ángel Rubén Barberis[1] and Lorena E. Del Moral Sachetti[2]
[1]*National University of Salta, Headquarters - Salta Capital – Argentina*
[2]*National University of Salta, Orán Regional headquarters – Salta – Argentina*

ABSTRACT

The optimization of Call Centers is not an easy problem to solve, due to the complexity of the mathematical models that derive from the Erlang formulas. This complexity is transferred to optimization models, which in most cases are made up of non-linear and non-differentiable objective functions. As in all areas of Operations Research, solving these problems demands efficient, fast and precise algorithms. Simulation as an experimental tool constitutes an essential environment for the validation of optimization algorithms, especially when there are no well-defined problem repositories with known results metrics that can be compared. This paper describes a strategy that uses stochastic simulation to study the statistical convergence of integer nonlinear optimization algorithms in the study of Call Center problems.

KEYWORDS

Convergence of Optimization Algorithms, Integer Nonlinear Optimization, Call Center Optimization

1. INTRODUCTION

Optimizing operational resources in Call Center problems is a difficult process to solve accurately, mainly due to the complexity of Erlang's models, especially when trying to approximate reality. Although the mathematical formulations for the Erlang-A queue models (M/M/*n* + G) are available, they are too complicated to derive analytical and algorithmic solutions for Call Center problems. Therefore, solving issues such as estimating the additional number of operators needed when the volume of incoming calls is doubled, or determining the sensitivity of the model when there are errors in the estimation of patience (Zeltyn et al., 2005), among others, is usually a difficult task.

The optimal Call Center administration mainly pursues two opposing objectives: 1) Sizing with the lowest cost in hiring personnel (fewer personnel, lower level of service); and 2) Maximize service levels, which translates into maximization of user satisfaction (higher service level implies a greater number of personnel to be hired). For the first objective, integer linear optimization problems are solved. In the literature, there is a wide development of efficient research and algorithms on the subject (Atlason et al., 2007, Caprara et al., 2003, Ingolfsson et al., 2010, Robbins et al., 2010). Achieving the second objective involves solving an integer non-linear optimization problem, for which there is no general algorithm that can be applied in search of the solution, so that specific algorithms must be developed. In the literature, works are recorded in which a joint approach is carried out, prioritizing the solution to the first objective, and secondly, they calculate an approximation to the second objective's solution (Ingolfsson et al., 2002, Koole et al., 2003). Mathematical drawbacks hinder the development of general analytical processes that can prove convergence of non-linear optimization algorithms as a consequence of the diversity of approaches with which the problem is addressed. The difficulty is even greater as it does not have, in this research area, well-defined problem repositories, with known results metrics with which it can be contrasted. For this reason, some researchers use simulation as a comparison tool, supported by confidence intervals (Kim et al., 2011).

In Call Center problems, the use of simulation is very common, since it is an excellent support tool in decision-making (Avramidis et al., 2005, Chokshi, 1999, Sencer et al., 2013). It is also, very commonly used in technological comparison of different scheduling approaches. (Gulati et al., 2001).

In order to have in our research a validation tool for non-linear optimization processes, statistics are resorted to in order to use from it, some concepts such as Residual Analysis, the Empirical Rule 68-95-99 and the evaluation of Kurtosis. All these combined with a simulation tool allow to specify a statistical method to evaluate the degree of convergence of a numerical series generated by an algorithm. This work describes a strategy to validate results and convergence obtained from optimization algorithms, called Residual Empirical Validation. This statistically compare the results obtained by an algorithm with those obtained by simulation, and under certain premises of statistical variability it can be concluded whether the algorithm is convergent or not. This procedure is described in section 2. Section 3 presents a practical case of convergence validation of an integer nonlinear optimization algorithm applied to the Call Center problem. Finally, the conclusions are presented in section 4.

2. RESIDUAL EMPIRICAL VALIDATION WITH SIMULATION

2.1 Residual Analysis

The *residual analysis* is the tool used to evaluate the suitability of a linear regression model against a set of experimental data by defining residuals and verifying statistical assumptions through the analysis of residual plots (Asuero et al., 1989, Cook et al., 1982, Topp et al., 2004).

The main statistical assumptions regarding the regression model that are verified with the residuals plot are (Black, 2010, Cook et al., 1982, Massart et al., 2003, Verran et al., 1984):

1. The generation of residuals must have a random behavior and be independent and identically distributed.
2. All residuals must have the same variance. This means that the variance must be constant throughout the dynamic concentration range of the residuals. This property is known as *homoscedasticity*.
3. All residuals must be random variables with (approximately) normal distribution with mean 0, therefore the expected value of residuals must be 0.

The verification of properties, such as being independent and identically distributed, leads to the fulfillment of other implicit assumptions (Korner-Nievergelt et al., 2015).

Once the residual *independence is* verified, it implies the fulfillment of assumptions such as: ***a)*** the residuals are not correlated with any other variable included or not in the model; ***b)*** The residuals are not grouped (i.e., the sample means of any set of residuals should all be equal); ***c)*** The residuals are not autocorrelated (i.e., no temporal or spatial autocorrelation exist).

On the other hand, to say that the residuals are *identically distributed* means that: ***a)*** All the residuals come from the same distribution. In the case of a linear regression, it is assumed that they all come from the same normal distribution; ***b)*** The residual variance is homogeneous, that is, the homoscedasticity property is fulfilled; ***c)*** The mean of the residuals is zero throughout the range of predictor values. When numerical predictors (covariates) are present, it implies that the relationship between the independent and dependent variable can be adequately described by a straight line in the plane.

The difference between the observed values of the dependent variable (y) and the values predicted by the model ($\hat{y}$) is called *residual* (e) (Martin et al., 2017). Also known as *error*. That is:

Residuals (e) = Observed values (y) – Values predicted by the model ($\hat{y}$) $\Rightarrow e = y - \hat{y}$

Where $\sum e_i = 0$ and $\overline{e} = 0$ provided that the statistical assumptions are fulfilled.

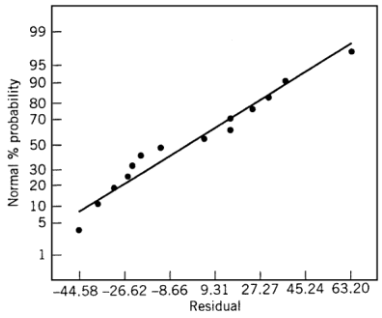

Figure 1. Residuals with normal distribution

A residual plot shows the residuals on the vertical axis and the independent variable on the horizontal axis. When the points on the graph are randomly scattered around the horizontal axis, the linear regression model is considered appropriate for the data. A quick way to check the assumption of normality is to analyze the trend of the graph. If the residuals are drawn approximately along or over the straight line as in figure 1, then the normality assumption is

fulfilled (Myers et al., 2009), which implies immediately satisfying the assumptions of implication mentioned above. From the conceptual point of view, the analysis of residuals constitutes a simple tool that facilitates the detection of errors and the verification of the degree of approximation between an empirical or theoretical equation with experimental results (Tomàs et al., 2006).

2.2 Empirical Rule or 68-95-99 Rule

Using a central estimate such as the mean or median along with a measure of variation (such as the standard deviation or interquartile range) in a sampling distribution is a good way to describe the values or behavior of a population. In case that the relative frequency histogram of the data has the shape of a bell (or has an approximately normal distribution), the population mean and the standard deviation are the appropriate combination to study variability or dispersion, and a special rule links them to obtain fairly detailed information about the general population. This rule is the so-called *empirical rule* , also known as the 68-95-99 rule (Rumsey, 2016).

The empirical rule is a general rule that is used to indicate the approximate percentage of sample values that are found within the interval formed by the standard deviations with respect to the sample mean, when these are normally distributed (Black, 2010). The rule is generally applied to a random variable that follows a normal distribution, with mean μ and standard deviation σ. An important characteristic expresses that, if the distribution of the data is more or less symmetric, unimodal and follows a normal law or an approximation to it, then approximately 68.27% of the data are concentrated within the range $\mu \pm \sigma$; 95.45% within $\mu \pm 2\ \sigma$, and 99.73% within $\mu \pm 3\ \sigma$. Figure 2 illustrates the three intervals of the 68-95-99 rule. An important point to keep in mind is that the empirical rule does not apply to data sets with very asymmetric distributions.

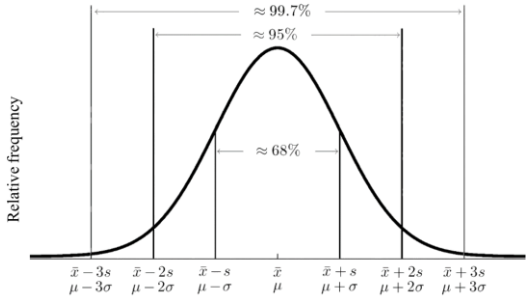

Figure 2. Empirical Rule. Percentage data according to distance from the mean

2.3 Simulation

The Call Center simulator developed in the research project executes a stochastic model that has been properly verified in (Barberis et al., 2011).

The special feature of this tool is that it implements a device that allows to dynamically carry out the pre-established shift assignment policy. This mechanism, not foreseen in the commercial Call Center simulation software, enables and disables a certain number of agents

according to the stipulated shift scheduling, as the simulation of a working day elapse. In this manner, the simulations that are carried out are very close to the operational reality of the Call Centers, which makes it possible to formulate satisfactory and precise conclusions regarding the objective reality.

The stochasticity of computer simulations depends on the generation of pseudo-random numbers. In successive simulation executions, the algorithmic generation of pseudo-random numbers produces results that violate the statistical assumptions of the residuals analysis, that is, correlated estimates are generated, which would not be independent nor identically distributed as random variables (Law et al., 2000). To guarantee the stochastic process and avoid the problems generated by pseudo-randomness, two alternatives can be resorted to generate samples that pass statistical tests of randomness and independence:

1) Perform n different executions of the simulation and extract from them a subset of m ($m < n$) results of the study variable, preferably those with the greatest variability. This will lead to m observations. Or,

2) Form n groups of m different simulation executions, with n different seeds for the generation of pseudo-random numbers (each group uses the same seed for the m simulations). Each group of m simulation results forms a row of the matrix of size $n \times m$. Then, a new random sample is designed by averaging the values of each column of the matrix, as shown in figure 3, thus generating m simulation observations.

From either alternative, m independent observations are obtained, identically distributed, and approximately normal, very convenient for the generation of residuals.

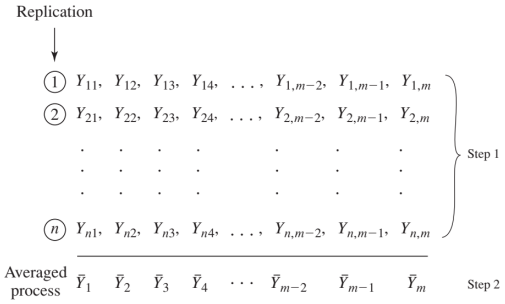

Figure 3. Sample formation process from simulations

2.4 Residual Empirical Validation Process

The objective is to know if the optimization algorithm executed with different initial points from the decision space converges to the same limit point, and if this is close to the solution of the problem under study or not. The conclusions are developed from statistical comparisons between the outputs of a simulation process and the deterministic results obtained by the objective algorithm.

The procedure consists in generating observations from the results of the algorithm which are obtained from m executions with m different starting points from the decision space. The simulation processes are also started with the m different points used by the algorithm, see the diagram in figure 4. If A_i is the observed value obtained as a result of the algorithm executed in the i-th experiment, and S_i the estimation of the same study variable obtained by simulation in

the *i*-th process, then the residues $r_i = a_i - S_i$, $\forall I = 1... m$, form the residual random variable *R*. Thus, if $A_i \rightarrow E(S_i)$ then $E(R) \rightarrow 0$. After generating the sample, the residuals analysis is performed to ensure fulfillment of the statistical assumptions. Failure to comply with some of the assumptions implies increasing simulation times and generating a new sample of simulation observations using the process of the alternative not chosen. If some of the statistical assumptions of the analysis of residuals were to fail again, then the hypothesis that the algorithm converges to the solution of the problem under study is rejected. In the case that all the assumptions are verified, it is considered that the algorithm tends to converge to a limit point that may or may not be close to the solution of the target problem. The next step is to analyze the degree of concentration of the residues around the central measure. The estimate is obtained from the Kurtosis indicator, which measures the degree of flatness or tailedness of the shape of the sample distribution with respect to the normal curve (Khurshid et al., 2007). In the comparison with the normal distribution curve, if the kurtosis indicator is greater than 0, the sample distribution will be Leptokurtic; if it is equal to 0 it will be Mesokurtic, (it indicates that it has a normal distribution); however, if the indicator is less than 0, the distribution will be Platykurtic (Chissom, 1970, Pearson, 1905). For a successful application of the empirical rule, a kurtosis indicator larger or equal to 0 is desirable. This will give rise not only to a successful validation of the algorithm's convergence, but also to a convergence with an acceptable degree of precision.

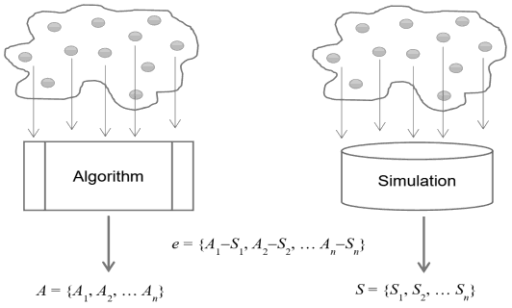

Figure 4. Residuals Sample formation process

Once the kurtosis condition is satisfied, the results derived from the Standard Deviation (σ) will be studied in its three values: σ, 2σ and 3σ, according to the *68-95-99 empirical rule*. It is desirable that 100% of the algorithmic results are within an acceptable approximation range around the solution. As it is a statistical study, it will be sought that at least a certain percentage of the algorithmic results have an acceptable precision, which implies setting an upper bound of 2σ or 3σ, which will depend on the type of problem to be solved. Thus, for example, for the case study described in section 3.2, which seeks to determine the optimal distribution policy of working shifts for the operators of a Call Center, which maximize service levels (SL), requires that 2σ or 3σ be less than 1 – since, the SL is quantized in the real range (0, 1] – to ensure that at least 95% of the algorithmic results are within the precision range.

3. APLICATION OF THE TECHNIQUE

3.1 The Problem to Solve

In the field of Call Centers, there are no algorithmic solutions that have clearly achieved optimal results; much less, there is consensus on the best implementation strategy. That is why the research that is developed it is about designing algorithmic alternatives that seek to optimize the planning and distribution policies of shifts, from the perspective of Integer Nonlinear Optimization. Thus, the problem to solve is of the form:

$$\begin{aligned} &máx \;\; f(x) \\ &s.a. \;\; Ax \geq r \; ; \\ &\qquad Bx = \Omega; \;\; x \geq 0; \; x \in \mathbb{Z}^n \end{aligned} \tag{1}$$

Where $f(x)$ is the non-linear objective function with domain over a discrete decision space, which measures the performance of the Call Center in terms of service level (SL). The function is non-differentiable, non-convex, and does not configure a quadratic problem. The constraints of the problem are linear convex and are composed of a matrix $A \in {}'^{m \times n}$, and vectors B, $r \in {}'^{m}$, and $\Omega \in {}'$.

Table 1. Pseudo-code of the algorithm that solves the problem (1)

Algorithm: Basic optimization procedure with directional search
0. `Let` $x^{(0)}$ an initial point of the decision space.
1. $x^{(k+1)} = x^{(0)}$
2. Update k-th iteration.
3. $x^{(k)} = x^{(k+1)}$
4. **`For`** $i = 0, 1, 2, \ldots \lvert D \rvert$
5. ⎸ `Let` d_i directional vector D.
6. ⎸ Calculate $\alpha^* = \arg máx\, h(\alpha) = F(x^{(k)} + \alpha \cdot d_i)$
7. ⎸ Update $x^{(k+1)} = x^{(k)} + \alpha^* d_i$
8. **`If`** $F(x^{(k+1)}) > F(x^{(k)})$ **`go back to step 2`**.
9. `Let` $x^a = x^{(k+1)}$
10. `Let` $x^{(k+1)} = Set\_to\_Integer\ (x^a)$
11. **`If`** $F(x^{(k+1)}) > F(x^a)$ **`go back to step 2`**.
12. **`Return`** $x^{(k+1)}$

The algorithm selected for the experiment is described in (Barberis et al., 2019) and has the particularity of being simple in its design and obtaining results at an acceptable speed. In the pseudo-code shown in table 1, $F(x)$ is a penalty function of the form $F(x) = f(x) + P(R(x))$ where $f(x)$ is the objective function, $R(x)$ constitutes the restrictions of (1) and $P(r)$ is the function that returns 0 if x satisfies the restrictions of the problem, and a negative weighted value if it does not satisfy them. It is also required to fit the values of the real components of $x^{(k+1)}$ to the nearest integer vector. The conversion task is carried out by the *Set_to_Integer* (v) function on line 10. This is a complex process that searches for the integral vector closest to the real vector that best evaluates $F(x)$.

3.2 Computational Results

The validation of the algorithm, aims to give reliability in stability and convergence towards a satisfactory result, considering certain precision. The method consists of 7 steps, of which the first 5 corresponds to the study of convergence towards a limit point; and the last 2, to the study of the proximity of the limit point to the solution of the optimization problem.

To have greater clarity in the experiments, the analysis for a problem of dimension 6 corresponding to data from a small Call Center, whose parameters and solution are known, is shown in detail. The parameters used are the same for both the algorithm and the simulator, and correspond to: a working day of 9 hours; 4-hour work shift; 1-hour observation period; 180 second AHT; 20 second AWT; average abandonment equal to 1.5; 60% call retries, and a target service level of 95%. With these data, algorithmic and simulation experiments are carried out.

1) *Ensure that the residual sample is independent and identically distributed*.

Table 2. Residuals from 10 experiments with different starting points

Experiment	X (starting)	SL-Algorithm	SL-Simulation	Residues
1	5, 1, 2, 2, 6, 3	0,988567835	0,988556579	1,1256E-05
2	5, 2, 1, 2, 6, 3	0,988595998	0,988556676	3,9322E-05
3	5, 3, 0, 5, 3, 3	0,988545779	0,988556877	-1,1097E-05
4	5, 4, 0, 7, 0, 3	0,988569574	0,988556239	1,3334E-05
5	5, 5, 3, 2, 1, 3	0,988555999	0,988557011	-1,0118E-06
6	5, 6, 2, 0, 3, 3	0,988556784	0,988555922	8,623E-07
7	5, 7, 2, 0, 2, 3	0,988549359	0,988556722	-7,3629E-06
8	5, 8, 1, 0, 2, 3	0,988565779	0,988556658	9,1206E-06
9	5, 9, 1, 1, 0, 3	0,988549496	0,988557104	-7,6083E-06
10	5, 5, 0, 6, 0, 3	0,988556779	0,988556848	-6,89E-08
			$\sum e$ =	4,6746E-05
			$\overline{e}$ =	4,6746E-06
			σ =	1,404909E-05
			Durbin-Watson=	2,21663062
			Kurtosis =	0,445003077

In the simulation experiment, 10 independent replications of 20 runs of a 5,000- workdays simulation are performed. The 10 runs with the greatest variability are extracted from each replication and their results are averaged. Table 2 shows the 10 observations obtained as results of the algorithm and the simulation experiment. The study variable is SL and represents the value of $f(x)$ in (1) when evaluated at some point of the decision space. The initial planning vector is shown in the table with column X, and represents the feasible point from which the approximation sequence must start. The comparison of the study variable SL is carried out between the deterministic model of the algorithm and the stochastic model of the simulation. The residual results are shown in the last column.

Assuming that the results given by the algorithm are comparable with those obtained by simulation, then the residual analysis must show that the assumptions 2 to 5 are fulfilled.

2) *The residuals are centered around the abscissa axis (to verify the assumption of normality)*.

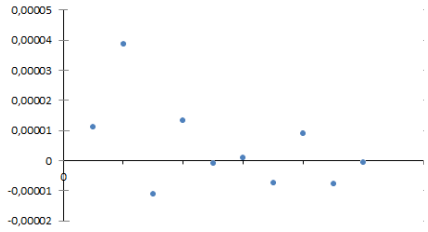

Figure 5. Residual plot

The assumption of normality is verified with the graph in figure 5, which shows the closeness and distribution of the sample residuals around the abscissa axis.

From the graphical analysis, the fulfillment of the normality assumption is deduced.

3) *The distribution of the residuals follows the normal law of probability (the sum of the residuals and the sample average must tend to zero)*.

Table 2 shows that $\sum e_i \to 0$ and $\overline{e} \to 0$, and after verifying the assumption of normality with the residuals graph, it is accepted that the sample has an approximately normal distribution with the sample mean tending to zero.

4) *The residuals do not present autocorrelation, that is, the sample is conformed of independent residuals amongst themselves*.

To verify that statistically there is no autocorrelation (they are independent), the Durbin-Watson indicator (Durbin et al., 1950), will be used , whose expression is shown in Figure 6, which takes a value of 2 when the residuals are completely independent.

$$DW = \frac{\sum_{i=2}^{n} \left(e_i - e_{i-1}\right)^2}{\sum_{i=1}^{n} e_i^2}; \quad 0 \le DW \le 4$$

Figure 6. Durbin-Watson indicator

A $DW < 2$ indicates positive autocorrelation, and a $DW > 2$ a negative autocorrelation. In practice when DW is between 1.5 and 2.5 it is considered that there is independence. To calculate this, the DW statistic is compared with critical values according to the level of significance α. These values are tabulated for the critical estimators d_L and d_U (Durbin et al., 1951). To determine the decision criterion, the DW statistic calculated in table 2 is considered. This is $DW = 2.21663062$. To obtain the critical values d_L and d_U with a significance level of 5%, the number of samples $n = 10$ and the number of independent variables $k = 1$ are necessary.

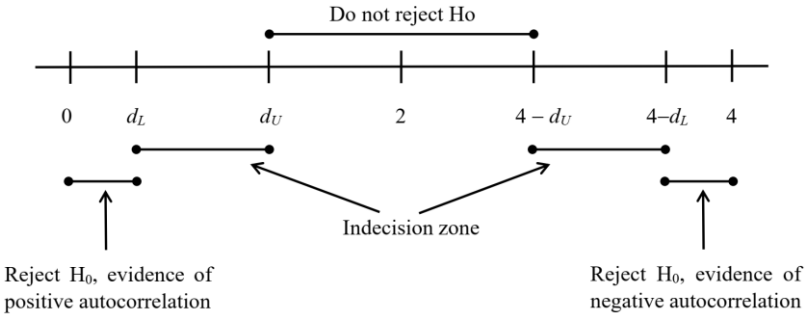

Figure 7. Decision Criterion according to (Welti, 2002)

The critical values obtained from the Durbin-Watson table with 5% significance are:

$$d_L = 0{,}879 \quad \Rightarrow \quad 4 - d_L = 3{,}121 \qquad\qquad d_U = 1{,}320 \quad \Rightarrow \quad 4 - d_U = 2{,}68$$

The null hypothesis H_0 is contrasted against the alternative H_a, where H_0 = *The residuals are not autocorrelated* and H_a = *The residuals are autocorrelated*. Figure 7 shows the non-rejection area of H_0. Since $DW \in [d_U\ ;\ 4 - d_U] = [1{,}320;\ 2{,}68]$ the null hypothesis H_0 is not rejected. With this, the assumption of independence between the sample residuals is verified.

5) *The dispersion of the residuals is constant (homoscedasticity).*

To demonstrate homoscedasticity, the *Bartlett Test* is used (Bartlett, 1937). This defines the null hypothesis, H_0 that the variances of *k* independent samples of a population are equal, versus the alternative hypothesis that at least two are different. The statistic is shown in Figure 8.

The test statistic *T* has a distribution which is approximately χ^2_{k-1}. Due to this, the null hypothesis is rejected if $T > \chi^2_{k-1,\alpha}$ with a significance level α.

$$T = \frac{(N-k)\ \ln(S_p^2) - \sum_{i=1}^{k}(n_i - 1)\ \ln(S_i^2)}{1 + \frac{1}{3(k-1)}\left\{\left(\sum_{i=1}^{k}\frac{1}{n_i - 1}\right) - \frac{1}{N-k}\right\}}$$

$$donde\ \ N = \sum_{i=1}^{k} n_i \quad ; \quad S_p^2 = \frac{\sum_{i=1}^{k}(n_i - 1)\ S_i^2}{N-k}$$

Figure 8. Bartlett statistic

From table 2, six different groups of five residual elements each are formed. The groups are shown in table 3.

The estimated Bartlett statistic is $T = 6.8778$, and the critical value with significance of 5% (which is usual) is $\chi^2_{5;\ 0.05} = 11{,}0705$. Since $T < \chi^2_{5;\ 0.05}$ then H_0 is not rejected, so the homoscedasticity assumption is verified and the normality assumption completely verified.

In this way, compliance with all the assumptions of the Residual Analysis is verified. Therefore, it can be concluded that the algorithm statistically converges to a limit point oriented towards the solution of the problem (1).

Table 3. Selection of different groups for the evaluation of homoscedasticity

	M_1	M_2	M_3	M_4	M_5	M_6
1	1,1256E-05	-1,10978E-05	1,1256E-05	3,93222E-05	1,1256E-05	3,93222E-05
2	3,93222E-05	-1,0118E-06	-1,1098E-05	1,33346E-05	-1,1098E-05	-1,0118E-06
3	1,33346E-05	-7,3629E-06	-1,0118E-06	8,623E-07	1,33346E-05	8,623E-07
4	8,623E-07	-7,6083E-06	-7,3629E-06	9,1206E-06	-7,3629E-06	-7,6083E-06
5	9,1206E-06	-6,89E-08	-7,6083E-06	-6,89E-08	-6,89E-08	9,1206E-06
Var(M_i)	**2,1064E-10**	**2,22167E-11**	**7,8224E-11**	**2,5624E-10**	**1,18635E-10**	**3,39505E-10**

6) *Kurtosis analysis*

The next step is to analyze the degree of accuracy of the convergence. To do this, the degree of tailedness of the approximately normal distribution of the residuals sample is measured, which is achieved with the kurtosis indicator. Table 2 shows the calculation of the kurtosis indicator according to (DeCarlo, 1997), whose expression is shown in Figure 9.

$$C = \frac{\sum\left(X_i - \bar{X}\right)^4 / n}{\left[\sum\left(X_i - \bar{X}\right)^2 / n\right]^2} - 3 \;=\; 0,445003077$$

Figure 9. DeCarlo kurtosis indicator calculated according to table 3

Being C > 0 it shows that it is a distribution with positive kurtosis of the leptokurtic kind, which is the desired one. On the other hand, some researchers state that the sample moments m_r are not unbiased estimates of the population moments μ_r, and propose another expression for the kurtosis indicator that is unbiased for normal distributions (Joanes et al., 1998). The expression is shown in Figure 10, where *S* denotes the standard deviation. Considering the residual sample in Table 2, the unbiased kurtosis estimator is $K = 2.878598261$, which is widely greater than zero ($K > 0$).

$$K = \left\{\frac{n(n+1)}{(n-1)(n-2)(n-3)}\sum\left(\frac{x_j - \bar{x}}{s}\right)^4\right\} - \frac{3(n-1)^2}{(n-2)(n-3)}$$

Figure 10. Joanes Kurtosis Indicator

Both kurtosis indicators prove that the sampling distribution is of the leptokurtic kind. This means that most of the residual observations are located in an environment close to the expected value $E(e) = 0$.

7) *Analysis by Empirical Rule* $\rightarrow$ *degree of precision*

Considering the previous result, and by application of empirical rule *68-95-99*, it is known that 99.7% of the observations are concentrated in the range either in [$\overline{x}$ -3σ; $\overline{x}$ $+3\sigma$] or in [μ -3σ; μ $+3\sigma$]. Therefore, to statistically estimate the degree of proximity of the limit point to the solution of the optimization problem, it is necessary to set an upper bound than 3σ. If the residual sample generates a value of 3σ greater than said bound, then it would be concluded that the limit point to which the algorithm converges does not have the desired proximity to the solution of the objective problem. Otherwise, the limit point is accepted as an approximate solution of the optimization problem. Given that SL – calculated through $f(x)$ in problem (1) – is a percentage indicator that is measured in the range (0, 1], then it is established that $3\sigma < 1$.

The case study that is developed in Table 2, shows that $\sigma = 0.00001404909$, so the value of $3\sigma = 0.00004214727$, being much less than 1. Thus, it is accepted that the limit point towards which the algorithm converges is close to the solution of the problem.

In the experiments carried out, a curious fact could be observed: the magnitude of the approximation error in the result of the algorithm is of the same order of magnitude as the standard deviation (σ). In the experiment $\sigma = 0.00001404909$, so approximately the final result of the algorithm has at least 4 digits of precision, which is considered very satisfactory.

4. CONCLUSIONS

The Residuals Analysis is a statistical process that allows to study the approximation characteristics in the regression models. In this work, said process is shown as a key resource in the convergence study of optimization algorithms. The combination of the kurtosis indicator with the 68-95-99 empirical rule made it possible to estimate the degree of similarity between the results obtained by the algorithm and those obtained by the simulation. The technique was also applied to larger problems, up to dimension 1000, using the same algorithm shown in section 3.1. The technique showed satisfactory and encouraging experimental results. Also, it allowed to analyze if the numerical sequence generated by the algorithm converges to a limit point, and if the latter, is within an acceptance range with respect to the solution of the optimization problem. It is risky and premature to draw strong and final conclusions of the overall applicability of the technique in all areas of optimizations in the field of Operational Research, so it is necessary to extend the study of applicability to other fields of knowledge.

As a disadvantage, the technique needs to have a solid reference framework and of recognized results to carry out statistical comparisons and obtain, from them, conclusive results. The Residual Empirical Validation technique could be a way to begin to discuss the convergence of optimization algorithms whose results are wrapped in a blanket of uncertainty.

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